

Review Article

On Fallacies in the Inductive Approach to Teaching Electromagnetic Theory

Burak Polat

Department of Electronics and Communications Engineering, Yildiz Technical University, Istanbul
Turkey, 0000-0002-3188-2670, abpolat@yildiz.edu.tr

(First received April 16, 2022 and in final form June 03, 2022)

Reference: Polat, B. On Fallacies in the Inductive Approach to Teaching Electromagnetic Theory. *The European Journal of Research and Development*, 2(2), 269–286.

Abstract

Certain of logical and analytical fallacies in the inductive approach to teaching classical electromagnetics - as adopted in countless textbooks with pedagogical concerns - are outlined and criticized. They include the concept of point charge, derivation of boundary conditions by integration, definitions of perfect electric/magnetic conductors and work in presence of static fields in stationary media. These concepts and descriptions are clarified in the context of Hertzian Electrodynamics which serves as the main frame for describing macroscopic electromagnetic phenomena. Proper definitions and theorems for perfect electric/magnetic conducting media are provided and the origin of the formulas for work in presence of static fields are highlighted with reference to postulates and field equations of Hertzian Electrodynamics.

Keywords: Electromagnetic theory, boundary relations, perfect conductor, Hertzian electrodynamics, Lorentz force, electrostatics, magnetostatics.

1. Introduction

Classical electromagnetism studies essentially all macroscopic relations between electric and magnetic sources and fields, as well as their connections to quantities from other disciplines of physics. As almost every technological advance in electrical engineering has its roots in the laws of electromagnetism, this encumbers professors of electromagnetic theory with the responsibility of questioning the scientific background of the inductive approach to field theory to raise the quality of education. I start my investigation in Section 2 by outlining certain of the logical and analytical fallacies in the inductive approach which is adopted in countless textbooks with pedagogical concerns. This is followed by a review of the axiomatic structure of Hertzian Electrodynamics (HE) which serves as the main frame for describing macroscopic electromagnetic phenomena. Finally, proper definitions and theorems for perfect electric and magnetic

conducting media are provided and the origin of the formulas for work in presence of static fields are highlighted with reference to postulates and field equations of HE.

2. An Outline of Certain Fallacies in Textbooks

2.1. Point Charge vs. Differential Charge

While a point is an essential topological element used in establishing the postulates of Euclidean geometry, its projection to advanced calculus is utterly useless. Point charges hanging in free space can be described as mappings that correspond a finite collection of coordinate points to a set of real numbers (charge values). Fundamental operations such as limit, derivative and integration do not make sense in such degenerate (dimensionless) domains in Riemann Calculus, whereas such maps are described as null sets with zero measure in Lebesgue integration theory. Accordingly, an introduction to field theory with a point charge leads into a dead end as one can establish a macroscopic theory only by a connection to Calculus of Infinitesimals. By the way, unavailability of transition from point charges to a continuous distribution is purely an analytical issue and should not be related to any physical discussion (such as nonconformity between particle and field theories). That being said, the only permissible way to reach at the expression of the total action of a material continuum at a distance is to start with defining the concept of differential charge

$$dq = \rho_f(\vec{r})d\mathcal{V} \text{ (Step \#1)}$$

via volume charge density function ρ_f and postulate its action at a distance as

$$d\vec{E}(\vec{r}) = \frac{dq}{4\pi\epsilon_0} \frac{\vec{R}}{R^3} \text{ (Step \#2).}$$

Here, $\vec{R} = \vec{r} - \vec{r}'$ is the well-known relative position vector extending from the source to observation point with norm $R = |\vec{R}|$. This requires to be followed by postulating field theory in the sense of Schwartz-Sobolev distributions so that the differential form of electric field, $d\vec{E}(\vec{r})$, becomes *equally informative* as its Coulomb integral representation

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\square^3} \rho_f(\vec{r}') \frac{\vec{R}}{R^3} d\vec{r}'^3 \text{ (Step \#3).}$$

This postulate also allows the calculation of the action of sources accumulated on nonvolumetric domains employing Dirac delta distributions. A point charge Q takes stage only now appearing as a fixed term in the volume density function

$$\rho_f(\vec{r}) = Q \delta_p \text{ (Step \#4),}$$

which describes “a total amount of Q charges accumulated on a point domain”. δ_p signifies the point distribution in \square^3 . It should be remarked that this representation does not validate the concept of a point charge Q as a source. The source is only (and always) represented by ρ_f and $Q = \int_{\square^3} \rho_f(\vec{r}') d\vec{r}'^3$ only appears as a fixed real number that corresponds to the volume integral of ρ_f . Now, the action of this canonical source distribution can be calculated by substituting ρ_f back into the Coulomb integral, which is already postulated to apply in the sense of distributions, to reach at its action at a distance as

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0} \frac{\vec{R}}{R^3} \text{ (Step \#5).}$$

While all concepts and equations mentioned so far are familiar to the general reader, their order of introduction is of vital importance from a scientific view.

The following well-known postulates (cf.[1,p.4,5], [2,p.12], [3,p.11]) on point charge Q should be interpreted as postulates on the idealized source distribution $\rho_f(\vec{r}) = Q \delta_p$.

Postulate 1: There exist two kinds of electric charges: a positive charge and a negative charge. The “plus” and “minus” signs rest on the fact that their effects tend to cancel one another.

Postulate 2: Electric charge is conserved; it cannot be created or destroyed; in the sense that whenever any positive charge appears, an equal amount of negative charge also appears.

The third postulate that follows may only serve in particle theories and is not enrolled while establishing the axiomatic structure of macroscopic theory:

Postulate 3: All charges are integral multiples of the electronic charge, which is measured as $e = -1.60 \cdot 10^{-19}$ [C].

2.2. Boundary Conditions

We observe that the requirement of the concept of differential charge comes along with the necessity of an introduction to Schwartz-Sobolev theory of distributions [4] at the very beginning of establishing a macroscopic field theory, in which the supports of the sources are assumed to be defined in (or on) canonical geometries. This brings jump discontinuities or singularities in the expressions of fundamental fields in its wake; and

as a result, the tools from the space of continuous functions make sense no more. An example to such heuristic attempts is the application of divergence (and Stokes's) theorem in a volumetric region called pillbox (and along a rectangular loop) to derive the jump conditions on an interface without justifying the applicability of integral theorems of Vector Calculus in presence of singular fields; in other words, in the sense of distributions. For the interested reader the necessary scientific tools required to validate such integration techniques in deriving boundary relations as well as the concept of *equal information* of differential and integral forms of field laws are demonstrated in [5].

2.3. Description of Perfect Conductor Media

The next deficiency in teaching field theory is the lack of proper definitions of perfect electric and magnetic conductor media in physics literature. Perfect electric conductors (PEC) are generally introduced simply as “a supply of unlimited charges” and described by outlining its various properties which I borrow from [1, Sect.2.5.1] as follows:

- $\vec{E} = \vec{0}$ inside a conductor.
- $\rho_f = 0$ inside a conductor.
- Any net charge resides on the surface.
- A conductor is an equipotential.
- \vec{E} is perpendicular to the surface, just outside a conductor.

First of all, these descriptions are restricted to electrostatic fields. That means you will need other descriptions as the external sources generate other types of fields, viz. magnetostatic or static/dynamic electromagnetic fields. The logical fallacy here is that no set of descriptions can replace a definition in scientific methodology! It actually works the opposite way: *all descriptions are bound to rely on a proper definition.*

In the plateau of electrical engineering textbooks, generally the same approach is adopted (as adapted generally from reference physics books) lacking an original perspective (cf.[6, Ch.3]). We meet PEC descriptions (cf.[7, p.11]) frequently as “a medium with infinite conductivity” ($\sigma \rightarrow \infty$) with reference to Ohm's law in simple conducting media introduced as $J_c = \sigma \vec{E}$. Such a weak statement is again prone to criticism in many respects as follows:

- A general definition should not base on some limiting value of a constitutive parameter. Then your “definition” will be medium dependent. A proper physical

definition should be directly connected to the fundamental physical quantities such as charge, force, energy, associated with conservation laws.

- The common phrase “sigma tends to infinity” makes sense only for simple media where sigma is a scalar quantity.
- What is the point in describing a hypothetical medium over a physical (real) conductivity model if you are going to consider the limiting case as the constitutive parameter tends to infinity? That means you do not use that model at all.
- A conductivity model is always associated with current flow mechanism. Drude model is such an example. Such models require either static or dynamic electromagnetic fields (for DC/AC conductivity calculations). This means you cannot define a PEC remaining inside the laws and definitions of electrostatics at all; which is another negative item pedagogically!

2.4. Description of Work in Presence of Static Fields

In mechanics, work is defined as the total energy transferred to an object when the object is moved over a distance along a given regular path/curve C by an external force $\vec{F}_{ext}(\vec{r};t)$. It is expressed by the line integral $W_{ext}(t) = \int_C \vec{F}_{ext}(\vec{r};t) \cdot d\vec{C}$. Obviously, this

fundamental quantity cannot be explained by remaining inside the laws for stationary media. The following imaginary discussion between a sophomore student **A** and a textbook author **B** might be helpful to present the situation:

A: Work in electrostatics! Isn't it an oxymoron talking about the potential energy stored on a system of charges upon their transfer from ambiguous locations to their final destination in the context of a sub-discipline of electromagnetism where the charges are postulated to be at rest at all instants of observation?

B: We are actually talking about the total energy collected by the time the assumed rest condition is reached.

A: Even so, how can you calculate that total energy spent until that rest condition is obtained while limiting yourself to the concepts of electrostatics?

B: We are aware that there is no material displacement within the postulates on electrostatics. However, we *hypothesize* that the expression of electric field in electrostatics is still correct when an electrical source moves from one point to another along an arbitrary path under Coulomb force, which reads

$$dW = dq \int_{P_1}^{P_2} \vec{E}(\vec{r}) \cdot d\vec{C} = -dq \int_{P_1}^{P_2} \text{grad}V(\vec{r}) \cdot d\vec{C} = -dq \int_{P_1}^{P_2} dV = -dq[V(P_2) - V(P_1)]$$

A: You just used the Coulomb's force law $d\vec{F}(\vec{r}) = dq\vec{E}(\vec{r})$ by hypothesizing an extension of its range of validity to moving charges. Why don't you mention this in the first place while introducing the postulates and laws of electrostatics? That would probably make your definition legitimate then. It would also explain the acceleration of a static charge under a static electric field, which you also teach in the context of electrostatics.

B: No, it still would not be legitimate in the context of electrostatics. It requires to refer to force in the context of electrodynamics of moving bodies. However, we follow an inductive approach starting with static fields for pedagogical reasons.

I hope this short dialog has been satisfactory for the reader to realize the logical fallacy and insufficiency in the inductive approach. I sincerely do not believe that the majority of authors really think long and hard about the matter. You will see the situation better upon the following quotation from one of the most popular books [8, Sect.3.11] in electrical engineering community:

".. we indicated that electric potential at a point in an electric field is the work required to bring a unit positive charge from infinity (at reference zero-potential) to that point. To bring a charge Q_2 (slowly, so that kinetic energy and radiation effects may be neglected) from infinity against the field of a charge Q_1 in free space to a distance R_{12} , the amount of work required is

$$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}} "$$

You may realize that the author is clearly disturbed with the necessity to refer to the mechanism of motion in describing work, which intuitively drives him to pronounce "slow motion", "neglected kinetic energy", "radiation effects" (which are beyond the axiomatic frame of electrostatics) to reassure the reader of the liability of the end results.

The situation with the descriptions of work and energy in magnetostatics is relatively more complicated in literature. The general method of derivation is to utilize Faraday's law (of dynamic fields) to reach at the expression of magnetic energy of complete circuits (cf. [1,Sect. 7.2.4], [9,Sect.2.14]), which is beyond the axiomatic frame of magnetostatics.

3. The Axiomatic Structure of Hertzian Electrodynamics of Moving Media

Above-mentioned evidences from literature constitute a small portion of all possible examples that demonstrate the incapability of the inductive approach in establishing macroscopic field theory. On the other hand, all pieces drop into place when we switch to the deductive approach that incorporates the concept of motion along with the distributional tools. This can be managed by introducing with the main frame in the first place. This is Hertzian Electrodynamics (HE) of moving bodies, which is established by Postulates 7 and 8 as the covering theory of electromagnetism of stationary media given by Postulates 4-6:

Postulate 4: Macroscopic electromagnetic phenomena of stationary continuous material media are governed by Maxwell's equations.

Postulate 5: The force acting on volumetric free charges and conduction currents at rest (ρ_f, \vec{J}_c) in Maxwell's field theory of stationary continuous media is described by Lorentz's force law, which we express for the force volume density field as

$$\vec{f}(\vec{r};t) = \frac{d\vec{F}}{d\mathcal{V}}(\vec{r};t) = \rho_f(\vec{r};t)\vec{E}(\vec{r};t) + \vec{J}_c(\vec{r};t) \times \vec{B}(\vec{r};t) \quad (1)$$

Postulate 6: The laws of Classical Electromagnetism and complementary laws of stationary continuous media are valid in the sense of Schwartz-Sobolev distributions.

Postulate 7: The laws of Classical Electromagnetism of stationary continuous media and complementary laws¹ are *frame indifferent*.

Principle of Material Frame Indifference (PMFI) mentioned inherently in Postulate 6 is common for all disciplines of Classical Continuum Mechanics. PMFI dictates that all observers of an event are in full agreement with

- i. the nature (or state) of any physical quantity
- ii. the structural form and content of any conservation law, and
- iii. the result of any measurement taken in any frames.

The projection of PMFI onto macroscopic electromagnetism is the Hertz-Heaviside field equations (HHFE)

$$\begin{cases} \text{curl}(\vec{E}(\vec{r};t) - \vec{v}(\vec{r};t) \times \vec{B}(\vec{r};t)) + \partial_t \vec{B}(\vec{r};t) = \vec{0} \\ \text{curl}(\vec{H}(\vec{r};t) + \vec{v}(\vec{r};t) \times \vec{D}(\vec{r};t)) - \partial_t \vec{D}(\vec{r};t) = \vec{J}_c(\vec{r};t) \\ \text{div} \vec{D}(\vec{r};t) = \rho_f(\vec{r};t) \\ \text{div} \vec{B}(\vec{r};t) = 0 \end{cases} \quad (2a-d)$$

¹ Complementary laws relate Classical Electromagnetism with other branches of physics (such as mechanics, thermodynamics, elasticity, acoustics, physiology, etc.) so that any physical model is uniquely constructed. Coulomb's and Lorentz's force laws, Joule's first law are among examples.

satisfied in arbitrary media moving with arbitrary velocity $\vec{v}(\vec{r};t)$ as observed in a stationary reference frame. HHFE dictate that instantaneous values of electromotive and magnetomotive forces (abbreviated as emf and mmf) induced around any closed contour as well as total electric and magnetic charges (fluxes) located (leaving) any arbitrary volumetric domain are *frame indifferent*. For a historic review and axiomatic foundations of HE I refer the reader to [10] and the references cited therein. Formulations of numerous canonical radiation and scattering problems in the context of HE by the present author are available in [11]-[22].

Postulate 8: The frame indifferent laws of Classical Electromagnetism and complementary laws are valid in the sense of Schwartz-Sobolev distributions.

4. Proper Definitions of Perfect Conductor Media

In this section I provide novel definitions for (stationary) perfect electric and magnetic conductor media based on Lorentz force in (1) and the line integral definition of work. Differential electric charge element can be expressed in either of the forms $dq_{\text{eff}}(\rho_f(\vec{r};t)d\mathcal{V}, \rho_s(\vec{r}_s;t)d\Sigma, \rho_L(\vec{r}_L;t)dC)$ and for our purpose I propose to describe its dual as *differential current element* $d\vec{J}$ [Am], by which the type of current and its direction of flow are specified simultaneously by $d\vec{J} = (\vec{J}_C(\vec{r};t)d\mathcal{V}, \vec{J}_S(\vec{r}_s;t)d\Sigma, I(t)d\vec{C})$ as illustrated in Fig.1.

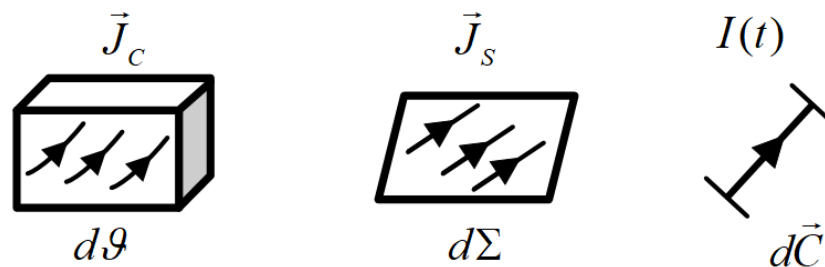


Figure 1 Differential current element $d\vec{J}$ in 3-D, 2-D and 1-D.

4.1 Perfect Electric Conductor

Definition 1: A Perfect Electric Conductor (PEC) is a hypothetical (volumetric) medium in which the total amount of work required to be done by external forces against the Lorentz force to carry a differential pair of electric source elements $(dq, d\vec{J})$ from a point P_1 to another point P_2 along an arbitrary path C as in Fig.2 is zero:

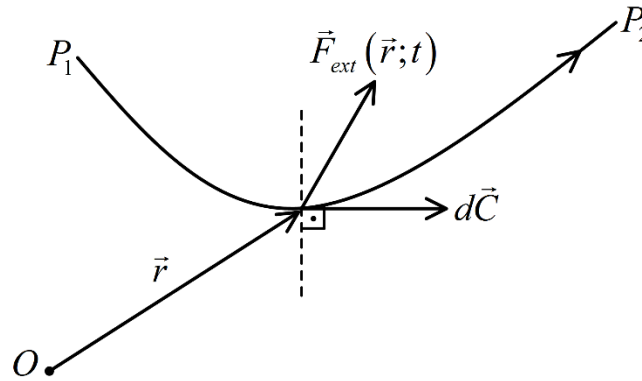


Figure 2 A point source traveling along an arbitrary path C .

$$dW_{ext}(t) = \int_C d\vec{F}_{ext}(\vec{r}; t) \cdot d\vec{C} = - \int_C (dq \vec{E} + d\vec{J} \times \vec{B}) \cdot d\vec{C} = 0 \quad (3)$$

Theorem of Localization in Appendix dictates that this relation is satisfied for any arbitrary path iff

$$\rho_f(\vec{r}; t) \vec{E}(\vec{r}; t) + \vec{J}_C(\vec{r}; t) \times \vec{B}(\vec{r}; t) = \vec{0} \quad (4)$$

at every point inside the PEC. I shall call (4) the *PEC condition in a volumetric region*.

Theorem 1: In a PEC medium one observes

$$\begin{cases} \rho_f(\vec{r}; t) = 0, \vec{E}(\vec{r}; t) = \vec{0}, \vec{D}(\vec{r}; t) = \epsilon_0 \vec{E}(\vec{r}; t) = \vec{0} \\ \vec{J}_C^e(\vec{r}; t) = \vec{0}, \vec{B}(\vec{r}; t) = \vec{B}_0 = \text{const} \end{cases} \quad (5)$$

Proof: To be able to understand field behavior in a PEC, we need to consider (5) along with Maxwell's equations in free space

$$\begin{cases} \text{curl } \vec{E}(\vec{r}; t) + \partial_t \vec{B}(\vec{r}; t) = \vec{0} \\ \text{curl } \vec{B}(\vec{r}; t) - c_0^{-2} \partial_t \vec{E}(\vec{r}; t) = \mu_0 \vec{J}_C(\vec{r}; t) \\ \text{div } \vec{E}(\vec{r}; t) = (1/\epsilon_0) \rho_f(\vec{r}; t), \text{div } \vec{B}(\vec{r}; t) = 0 \end{cases}$$

Since we are discussing a hypothetical medium, say \mathcal{G}_s , it is pointless to assign a material constitution other than (ϵ_0, μ_0) of free space with $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$. You may realize the geometric vector relations $\vec{E} \square \vec{J}_C$ and $\vec{J}_C \times \vec{B} \perp \vec{J}_C$. Therefore, (4) decomposes into

$$\rho_f(\vec{r}; t) \vec{E}(\vec{r}; t) = \vec{0}, \vec{J}_C(\vec{r}; t) \times \vec{B}(\vec{r}; t) = \vec{0} \quad (6a,b)$$

simultaneously. The condition (6a) can be satisfied iff $\rho_f(\vec{r};t)=0$ and $\vec{E}(\vec{r};t)=\vec{0}$. This can be seen as follows:

- Assume $\rho_f(\vec{r};t) \neq 0$, $\vec{r} \in \mathcal{G}_s$. Then, (3) reduces into $\int_{C \in \mathcal{G}_s} \vec{E}(\vec{r};t) \cdot d\vec{C} = 0$ for any arbitrary path C in the volumetric source region \mathcal{G}_s . This necessitates $\vec{E}(\vec{r};t) = \vec{0}$, $\vec{r} \in \mathcal{G}_s$. Inserting this result into Gauss' law one gets $\rho_f(\vec{r};t) = \epsilon_0 \text{div} \vec{E}(\vec{r};t) = 0$, $\vec{r} \in \mathcal{G}_s$, which contradicts with the initial assumption. Therefore, $\rho_f(\vec{r};t) = 0$, $\vec{r} \in \mathcal{G}_s$.
- Assume $\vec{E}(\vec{r};t) \neq \vec{0}$, $\vec{r} \in \mathcal{G}_s$. Then, one gets $\int_{C \in \mathcal{G}_s} \vec{E}(\vec{r};t) \cdot d\vec{C} \neq 0$ for any arbitrary path C . This contradicts with (3). Therefore, it reads

$$\rho_f(\vec{r};t) = 0 \text{ and } \vec{E}(\vec{r};t) = \vec{0}, \vec{r} \in \mathcal{G}_s.$$

Now that our investigation reduced into magnetostatic condition in the PEC, (6b) simplifies into $\vec{J}_c(\vec{r}) \times \vec{B}(\vec{r}) = \vec{0}$. This condition is satisfied iff $\vec{J}_c(\vec{r}) = \vec{0}$ and $\vec{B}(\vec{r}) = \vec{B}_0 = \text{const}$. Its demonstration follows a similar path:

- Assume $\vec{J}_c(\vec{r}) \neq \vec{0}$, $\vec{r} \in \mathcal{G}_s$. This condition imposes $\vec{B}(\vec{r}) \times \text{curl} \vec{B}(\vec{r}) = \vec{0}$, which is impossible for nonuniform fields. Therefore, one has $\vec{J}_c(\vec{r}) = \vec{0}$ and $\vec{B}(\vec{r}) = \vec{B}_0 = \text{const}$.
- Assume $\vec{B}(\vec{r}) \neq \vec{0}$, $\vec{r} \in \mathcal{G}_s$. This requires either $\vec{J}_c(\vec{r}) = \vec{0}$ or $\vec{B}(\vec{r}) \times \text{curl} \vec{B}(\vec{r}) = \vec{0}$. If $\vec{J}_c(\vec{r}) = \vec{0}$, this requires $\vec{B}(\vec{r}) = \vec{B}_0 = \text{const}$. Otherwise the condition $\vec{B}(\vec{r}) \times \text{curl} \vec{B}(\vec{r}) = \vec{0}$ is impossible for nonuniform fields. It requires $\vec{B}(\vec{r}) = \vec{B}_0 = \text{const}$ and therefore reveals $\vec{J}_c(\vec{r}) = \vec{0}$. This completes the proof.

The result $\vec{B}(\vec{r}) = \vec{B}_0 = \text{const}$ signifies that while dynamic magnetic fields cannot enter into a PEC medium, uniform streamlines of magnetic sources are unaffected by its presence. Since PEC, as a volumetric region \mathcal{G}_s , has the capability to hold free charges, it should be understood that free charges are accumulated on $\partial \mathcal{G}_s$, the enclosure (boundary surface) of \mathcal{G}_s .

How about the Lorenz potentials (V, \vec{A}) in a PEC medium? We observe that a nonuniform vector potential field which satisfies $\text{curl} \vec{A}(\vec{r}) = \vec{B}_0$ might exist in a PEC medium. On the other hand, the condition $\partial_t \vec{A} = \vec{0}$ reveals

$V(P_1) - V(P_2) = -\int_{P_1}^{P_2} \vec{E}(\vec{r}) \cdot d\vec{C} = 0, \forall P_{1,2} \in \mathcal{G}_s$. This signifies that all points in \mathcal{G}_s have the same electrostatic potential, which is equal to its value on $\partial\mathcal{G}_s$.

In the special case when PEC is a regular surface Σ as in Fig.3a we have

$$dW_{ext} = -dq \int_{C \in \Sigma} \vec{E}(\vec{r}; t) \cdot d\vec{C} = 0$$

with $dq = \rho_s d\Sigma$. Assume $\rho_s(\vec{r}; t) \neq 0, \vec{r} \in \Sigma$. Then,

$$dW_{ext} = -dq \int_{C \in \Sigma} \vec{E}(\vec{r}; t) \cdot d\vec{C} = 0$$

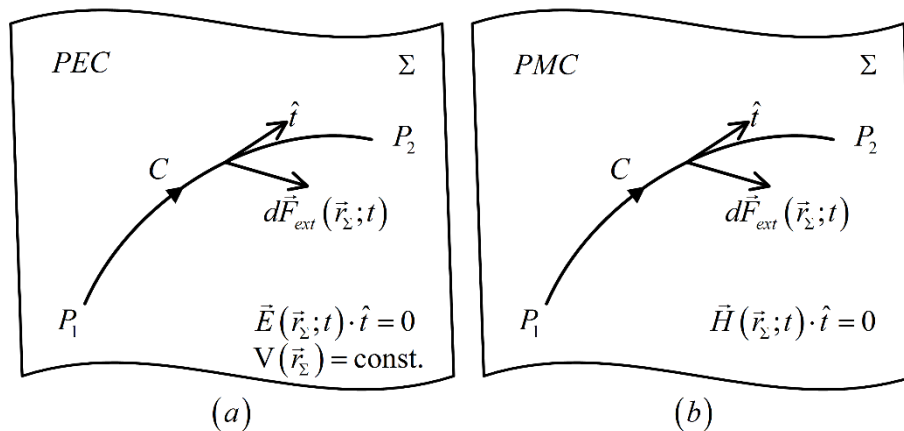


Figure 3 Boundary conditions on a (a) PEC (b) PMC surface

for any arbitrary path C lying on Σ . This necessitates $\vec{E}(\vec{r}; t) \cdot \hat{t} = \vec{0}, \vec{r} \in \Sigma$, where $d\vec{C} = \hat{t} dC$ with \hat{t} representing all vectors tangent to surface Σ . In electrostatics, this also reads $V(P_1) - V(P_2) = -\int_{P_1}^{P_2} \vec{E}(\vec{r}) \cdot d\vec{C} = 0, \forall P_{1,2} \in \Sigma$. We observe that the components of total electric fields tangential to a PEC surface are zero: $\vec{E}(\vec{r}_s; t) \cdot \hat{t} = 0, \vec{D}(\vec{r}_s; t) \cdot \hat{t} = 0$. On the other hand, there is no reason to assume $\rho_s(\vec{r}_s; t) = 0$. A body that contains no free charges has no influence on shaping of the static electric field in free space; which means it should be treated invisible electrically.

4.2 Perfect Magnetic Conductor

Definition 2: A Perfect Magnetic Conductor (PMC) is a hypothetical medium in which the total amount of work required to be done by external forces to carry a differential pair of magnetic source elements $(dq^m, d\vec{J}^m)$ from a point P_1 to another point P_2 along an arbitrary path C as in Fig.2 is zero:

$$dW_{ext}(t) = \int_C d\vec{F}_{ext}(\vec{r};t) \cdot d\vec{C} = -\int_C (dq^m \vec{H} - d\vec{J}^m \times \vec{D}) \cdot d\vec{C} = 0.$$

The dual sources are represented by $dq^m \mathfrak{E}(\rho_f^m(\vec{r};t)d\mathcal{V}, \rho_s^m(\vec{r}_\Sigma;t)d\Sigma, \rho_L^m(\vec{r}_L;t)dC)$ and $d\vec{J}^m = (\vec{J}_C^m(\vec{r};t)d\mathcal{V}, \vec{J}_S^m(\vec{r}_\Sigma;t)d\Sigma, I^m(t)d\vec{C})$. All relations for PMC can be obtained as dual transformation of those for PEC. The *PMC Condition in a volumetric region* reads

$$\rho_f^m(\vec{r};t)\vec{H}(\vec{r};t) - \vec{J}_C^m(\vec{r};t) \times \vec{D}(\vec{r};t) = \vec{0}. \quad (7)$$

Theorem 2: In a PMC medium one observes

$$\begin{cases} \rho_f^m(\vec{r};t) = 0, \vec{H}(\vec{r};t) = \vec{0} \\ \vec{B}(\vec{r};t) = \mu_0 \vec{H}(\vec{r};t) = \vec{0}, \vec{J}_C^{e,m}(\vec{r};t) = \vec{0} \end{cases} \quad (8)$$

This result signifies that *while dynamic electric fields cannot enter into a PMC medium, the streamlines of static electric sources are unaffected by its presence.*

In the special case when PMC is a regular surface Σ as in Fig.3b we observe that *the components of total tangential magnetic fields are zero: $\vec{H}(\vec{r}_\Sigma;t) \cdot \hat{t} = 0$, $\vec{B}(\vec{r}_\Sigma;t) \cdot \hat{t} = 0$.* On the other hand, there is no reason to assume $\rho_s^m(\vec{r}_\Sigma;t) = 0$. A body that contains no free magnetic charges has no influence on the shaping of the static magnetic field in free space; which means it should be treated *invisible magnetically*.

5. Origin of Work in Presence of Static Fields

Postulate 7 asserts that the total amount of work required to be done by external forces against the Lorentz force to carry a differential pair of electric source elements $(dq, d\vec{J})$ from a point P_1 to another point P_2 along an arbitrary path C as in Fig.2 is expressed by

$$dW_{ext}(t) = -\int_C (dq \vec{E} + d\vec{J} \times \vec{B}) \cdot d\vec{C}. \quad (9)$$

It can be seen that the instantaneous linear velocity of the differential sources – though they exist – do not appear explicitly in the formula.

5.1 Electrostatic Field

In the special case when the moving differential sources and the fields exerted upon them by exterior sources are both electrostatic, one can insert $\vec{E}(\vec{r};t) = \vec{E}(\vec{r})$, $\vec{B}(\vec{r};t) = \vec{0}$, $\vec{J}_c(\vec{r};t) = \vec{0}$ so that (9) reduces into

$$dW_{ext} = -dq \int_C \vec{E}(\vec{r}) \cdot d\vec{C} \quad (10)$$

The result applies for arbitrary values of the velocity of travel. Please be careful that this is a moving media formula, not an electrostatics problem! So, we refer to (2a) which simplifies into $\text{curl}\vec{E}(\vec{r}) = \vec{0}$ regardless of the velocity of motion. This irrotational property of static electric field shapes (10) into

$$dW_{ext} = dq \int_C \text{grad}V(\vec{r}) \cdot d\vec{C} = dq[V(P_2) - V(P_1)]$$

As a result, you can see that we have arrived at the formula, which is used without justification in typical textbooks.

5.2 Magnetostatic Field

In magnetostatics the differential current elements are static and appear only in the special forms $d\vec{J} = (\vec{J}_s(\vec{r}_2) d\Sigma, I d\vec{C})$. Accordingly, (9) reduces into

$$dW_{ext} = - \int_C (d\vec{J} \times \vec{B}(\vec{r})) \cdot d\vec{C}. \quad (11a)$$

When the particle is moving in the same direction with current flow, as is the case in motion along a conducting wire, one has $d\vec{J} \parallel d\vec{C}$ and $(d\vec{J} \times \vec{B}) \cdot d\vec{C} = 0$. We observe that the presence of magnetic field has no effect/role in the amount of total external force to be exerted. Then (11a) can be calculated as

$$dW_{ext} = - [d\vec{J}(\vec{r}_2) \cdot \vec{A}(\vec{r}_2) - d\vec{J}(\vec{r}_1) \cdot \vec{A}(\vec{r}_1)]. \quad (11b)$$

It is observed that the conditions $dW_{ext} > 0$, $dW_{ext} = 0$ and $dW_{ext} < 0$ are dependent on both the values of vector potentials and the orientations of the current source at initial and end points.

Under the condition $\vec{A}(\vec{r}) \rightarrow \vec{0}$, $r \rightarrow \infty$, we can assume point P_1 to be at infinity so that (11b) reduces into $dW_{ext} = -d\vec{J}(\vec{r}_2) \cdot \vec{A}(\vec{r}_2)$.

Finally, we may demonstrate (11b) for two possible cases of differential current element $d\vec{J} = (\vec{J}_s(\vec{r}_2) d\Sigma, I d\vec{C})$ separately as illustrated in Fig.4.

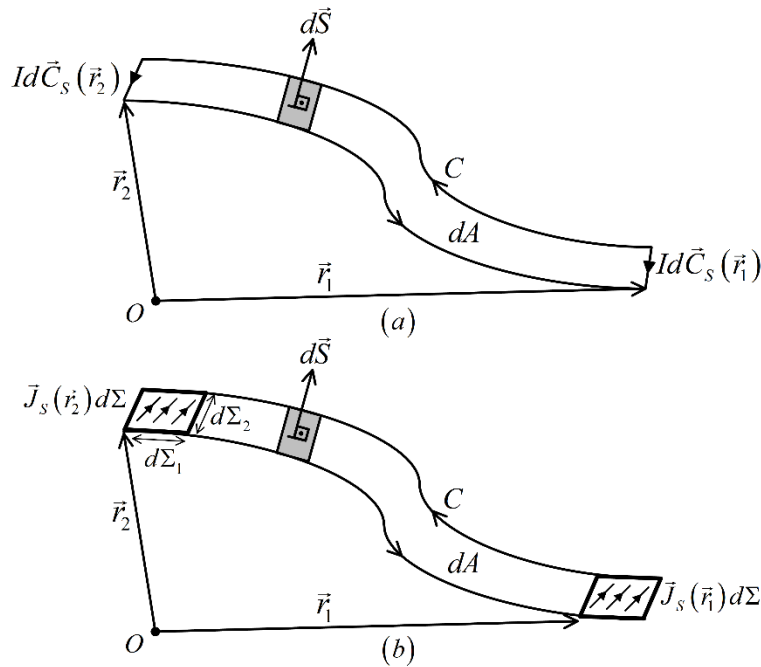


Figure 4 Differential (a) line (b) surface current element traveling along an arbitrary path C .

Case I: $d\vec{J} = I d\vec{C}_s$

$$\begin{aligned} dW_{ext} &= -I \int_C (d\vec{C}_s \times \vec{B}(\vec{r})) \cdot d\vec{C} = -I \int_C \vec{B}(\vec{r}) \cdot (d\vec{C} \times d\vec{C}_s) \\ &= -I \int_{dA} \vec{B}(\vec{r}) \cdot d\vec{S} = -I \int_{dA} \text{curl} \vec{A}(\vec{r}) \cdot d\vec{S} = -I \oint_{\partial dA} \vec{A}(\vec{r}) \cdot d\vec{l} \\ &= -[I d\vec{C}_s(\vec{r}_2) \cdot \vec{A}(\vec{r}_2) - I d\vec{C}_s(\vec{r}_1) \cdot \vec{A}(\vec{r}_1)] \end{aligned}$$

Here, the field relation $\vec{B}(\vec{r}) = \text{curl} \vec{A}(\vec{r})$ is a feature of Hertz-Heaviside field equation (2d). ∂dA signifies the enclosure (closed curve enclosing) the surface dA .

Case II: $d\vec{J} = \vec{J}_s(\vec{r}_\Sigma)d\Sigma$

$$\begin{aligned} dW_{ext} &= -\int_C \left(d\Sigma \vec{J}_s(\vec{r}_\Sigma) \times \vec{B}(\vec{r}) \right) \cdot d\vec{C} = -\int_C \vec{B}(\vec{r}) \cdot \left(d\vec{C} \times \vec{J}_s(\vec{r}_\Sigma) d\Sigma \right) \\ &= -J_s(\vec{r}_\Sigma) d\Sigma_1 \int_{dA} \vec{B}(\vec{r}) \cdot d\vec{S} \\ &= -J_s(\vec{r}_\Sigma) d\Sigma_1 \int_{dA} \text{curl} \vec{A}(\vec{r}) \cdot d\vec{S} = -J_s(\vec{r}_\Sigma) d\Sigma_1 \left[\oint_{\partial dA} \vec{A}(\vec{r}) \cdot d\vec{l} \right] \\ &= -\left[d\Sigma \vec{J}_s(\vec{r}_2) \cdot \vec{A}(\vec{r}_2) - d\Sigma \vec{J}_s(\vec{r}_1) \cdot \vec{A}(\vec{r}_1) \right] \end{aligned}$$

Dual transformations apply in calculating work in presence of a differential pair of magnetic source elements.

6. Conclusion

The present work contributes to literature by proper definitions and theorems for perfect electric/ magnetic conducting media and clarification of the origin of the formulas for work in presence of static fields in the context of Hertzian Electrodynamics. Also, the concept of *differential electric/magnetic current element* is introduced.

The thesis is that a scientific introduction to field theory requires the deductive approach of referring to the axiomatic structure of Hertzian Electrodynamics in the first place. In that context pedagogical efforts should rather focus on successful presentations of the deductive approach. Though it is not possible to grasp all details at once, still this gives a general idea to sophomore students dealing with the special case of static fields about where they stand so that they can associate each definition and mechanism they encounter along the inductive path to their origins in the general frame.

7. Acknowledge

This work was supported by Research Fund of the Yildiz Technical University. Project Number: FBA-2021-4310

Appendix:

Theorem of Localization: Let $\Phi(\vec{r})$ be a continuous scalar or vector function defined in some arbitrary domain Ω_N in \square^N . Then, there is one-to-one correspondence between the relations

$$\Phi(\vec{r}) = 0 \quad \text{and} \quad \int \cdots \int_{\Omega_N} \Phi(\vec{r}) dx_1 \cdots dx_N = 0 \quad (\text{i,ii})$$

to hold in every Ω_N .

Proof: If $\Phi(\vec{r}) = 0$, then this clearly requires $\int \cdots \int_{\Omega_N} 0 dx_1 \cdots dx_N = 0$. For the second step let us temporarily assume that $\Phi(\vec{r})$ is a scalar function which is different from zero, say positive, at some point P_o at $\vec{r}^o = (x_1^o, \dots, x_N^o)$. Then, due to continuity, it would also be positive in an ε -neighborhood $B\{\vec{r}^o, \varepsilon\} = \{(x_1, \dots, x_N) \mid 0 \leq |\vec{r} - \vec{r}^o| < \varepsilon\}$ of P_o so that the integral taken over $B\{\vec{r}^o, \varepsilon\}$, $\int \cdots \int_{B\{\vec{r}^o, \varepsilon\}} \Phi(\vec{r}) dx_1 \cdots dx_N$, would be non-zero (positive), which contradicts the assumption (ii). This completes the proof.

I adapted this theorem from a lemma by Sobolev [23,p.3]. It can also be extended to a vector field in a straightforward manner by treating each component separately. I dropped time dependence for simplicity as it does not violate generality.

References

- [1] Griffiths, D.J., (1989) Introduction to Electrodynamics, Prentice Hall, Inc.
- [2] Fano, R.M., Chu L.J., & Adler, R.B. (1960). Electromagnetic Fields, Energy, and Forces, John Wiley & Sons, Inc.
- [3] Magid, L.M., (1972). Electromagnetic Fields, Energy and Waves, John Wiley and Sons, Inc.
- [4] Schwartz, L. (1959). Theorie des Distributions, 2nd Ed. Paris Hermann, 1-2.
- [5] Polat, B., (2020). On Equivalence of Maxwell's Equations in Differential and Integral Forms, 2020 International Conference on Electrical, Communication, and Computer Engineering (ICECCE), Istanbul, Turkey, 1-6, doi: 10.1109/ICECCE49384.2020.9179294.
- [6] Elliot, R.S., (1993). Electromagnetics. History, Theory, and Applications, IEEE Press.
- [7] Jones, D.S. , (1964). The Theory of Electromagnetism, New York, Pergamon Press.
- [8] Cheng, D.K. (1983). Field and Wave Electromagnetics, Addison-Wesley Publ. Company, Inc.
- [9] Stratton, J.A., (1941). Electromagnetic Theory, McGraw Hill.
- [10] Polat, B., (2012). On the Axiomatic Structure of Hertzian Electrodynamics, TWMS Journal of Applied and Engineering Mathematics, 2(1), 17-41.
- [11] Polat, B. (2012). Scattering by a Moving PEC Plane and a Dielectric Half-Space in Hertzian Electrodynamics, TWMS Journal of Applied and Engineering Mathematics, 2(2), 123-144.
- [12] Polat, B. (2012). Scattering by a Moving Circular Cylinder in Hertzian Electrodynamics," Selçuk Journal of Applied Mathematics, 13(1), 89-109.
- [13] Polat, B. & Daşbaşı, R. (2019). Validation of Hertzian Electromagnetism in a Rectangular Waveguide with Rotating PEC Termination", 2019 Photonics & Electromagnetics Research Symposium – Spring (PIERS-Spring), Rome, Italy, 2850-2856. doi: 10.1109/PIERS-Spring46901.2019.9017394
- [14] Polat, B. & Daşbaşı, R., (2019). Plane Wave Reflection by a PEC Plane in Harmonic Motion, 2020 International Conference on Electrical, Communication, and Computer Engineering (ICECCE), Istanbul, Turkey, 1-6, doi: 10.1109/ICECCE49384.2020.9179436.
- [15] Polat, B. & Daşbaşı, R., (2020). "On Conservation of Electromotive Force in Hertzian Electrodynamics, Proceedings of 2nd International Conference on Electrical, Communication and Computer Engineering, ICECCE , Istanbul, TURKEY. doi: 10.1109/ICECCE49384.2020.9179200.
- [16] Polat, B. & Daşbaşı, R., (2020). "Hertzian Formulation of Scattering by Moving PEC Bodies," 2020 IEEE Ukrainian Microwave Week (UkrMW), Kharkiv, Ukraine, 486-493, doi: 10.1109/UkrMW49653.2020.9252730.

- [17] Daşbaşı, R. & Polat, B., (2020) "Plane Wave Scattering by a PEC Half-Plane in Uniform Rectilinear Motion," Progress In Electromagnetics Research B, 89, 111-132, doi: 10.2528/PIERB20061004.
- [18] Daşbaşı, R. & Polat, B., (2021) "An Analytical Approach to Rotor Blade Modulation", Wave Motion Vol.125 pp.1-25 (2021) DOI:10.1016/j.wavemoti.2021.102762.
- [19] Polat, B. & Daşbaşı, R., (2021) "Doppler Analysis of Dipole Antennas in Arbitrary Motion", Proceedings of 20th International Conference on Microwave Techniques COMITE 2021, Brno, Czech Republic April 19-21, 2021, pp. 1-6, DOI: 10.1109/COMITE52242.2021.9419876.
- [20] Polat, B. & Daşbaşı, R., (2021) "Free Space Doppler Analysis and RCS of a Moving PEC Plate Under Physical Optics Approximation" Proceedings of 8th International Conference on Electrical and Electronics Engineering (ICEEE 2021) Antalya, Turkey April 9-11, 2021, pp.27-31. DOI: 10.1109/ICEEE52452.2021.9415919.
- [21] Polat, B. & Daşbaşı, R., (2021) "Perfect Conductor Bodies of Revolution in Hertzian Electrodynamics" Proceedings of 8th International Conference on Electrical and Electronics Engineering (ICEEE 2021) Antalya, Turkey April 9-11, 2021, p.215-219. DOI: 10.1109/ICEEE52452.2021.9415970.
- [22] Polat, B. & Daşbaşı, R., (2022) "A Distributional Investigation of Hertz-Heaviside Field Equations" IEEE Transactions on Antennas and Propagation (2022) DOI:10.1109/TAP.2022.3140496
- [23] Sobolev, S. L., (1964). Partial Differential Equations of Mathematical Physics, Dover Publications.